

# BLACK HOLE ENTROPY

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## 1 Black-Hole Entropy Problem

According to the thermodynamical analogy in black hole physics, the entropy of a black hole in the Einstein theory of gravity equals  $S^{BH} = A_H/(4l_P^2)$ , where  $A_H$  is the area of a black hole surface and  $l_P = (\hbar G/c^3)^{1/2}$  is the Planck length [1, 2]. In black hole physics the Bekenstein-Hawking entropy  $S^{BH}$  plays essentially the same role as in the usual thermodynamics. In particular it allows one to estimate what part of the internal energy of a black hole can be transformed into work. Four laws of black hole physics which form the basis in the thermodynamical analogy were formulated in [3]. The generalized second law [1, 2, 4] (see also [5, 6, 7, 8] and references therein) implies that when a black hole is a part of the thermodynamical system the total entropy (i.e. the sum of the entropy of a black hole and the entropy of the surrounding matter) does not decrease. The success of the thermodynamical analogy in black hole physics allows one to hope that this analogy may be even deeper and it is possible to develop statistical-mechanical foundation of black hole thermodynamics.

Thermodynamical and statistical-mechanical definitions of the entropy are logically different. *Thermodynamical entropy*  $S^{TD}$  is defined by the response of the free energy  $F$  of the system on the change of its temperature:

$$dF = -S^{TD}dT. \quad (1)$$

(This definition applied to a black hole determines its Bekenstein-Hawking entropy.)

*Statistical-mechanical entropy*  $S^{SM}$  is defined as

$$S^{SM} = -\text{Tr}(\hat{\rho} \ln \hat{\rho}), \quad (2)$$

where  $\hat{\rho}$  is the density matrix describing the internal state of the system under consideration. It is also possible to introduce the *informational entropy*  $S^I$  by counting

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different possibilities to prepare a system in a final state with given macroscopical parameters from different initial states

$$S^I = - \sum_n p_n \ln p_n, \quad (3)$$

with  $p_n$  being the probabilities of different initial states. In standard case all three definitions give the same answer.

Is the analogy between black holes thermodynamics and the 'standard' thermodynamics complete? Do there exist internal degrees of freedom of a black hole which are responsible for its entropy? Is it possible to apply the statistical-mechanical and informational definitions of the entropy to black holes and how are they related with the Bekenstein-Hawking entropy? These are the questions which are to be answered.

Historically first attempts of the statistical-mechanical foundation of the entropy of a black hole were connected with the informational approach [2, 9]. According to this approach the black hole entropy is interpreted as "the logarithm of the number of quantum mechanically distinct ways that the hole could have been made" [9, 5]. The so defined informational entropy of a black hole is simply related with the amount of information lost by stretching the horizon, and as was shown by Thorne and Zurek it is equal to the Bekenstein-Hawking entropy [9, 5].

The dynamical origin of the entropy of a black hole and the relation between the statistical-mechanical and Bekenstein-Hawking entropy have remained unclear. In the present talk I describe some new results obtained in this direction.

## 2 Dynamical Degrees of Freedom

The problem of the dynamical origin of the black hole entropy was intensively discussed recently. The basic idea which was proposed is to relate the dynamical degrees of freedom of a black hole with its quantum excitations. This idea has different realizations [15, 16, 17, 18, 19, 20]

<sup>1</sup>. Here I discuss the recent proposal [16, 20] to identify the dynamical degrees of freedom of a black hole with the states of all fields (including the gravitational one) which are propagating inside the black hole.

In order to specify the corresponding internal states consider at first the process of quantum particles creation by the gravitational field of a black hole. This process can be described as the effect of parametric excitations of zero-point fluctuations propagating in the time-dependent gravitational field of a black hole. Particles are created in pairs. The creation of a particle outside the event horizon is necessarily accompanied by a creation of another particle inside the horizon. The latter particle has negative total energy<sup>2</sup>. As the result of the Hawking process of pairs creation

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<sup>1</sup> For recent review of the problem of the dynamical origin of the entropy of a black hole, see [21].

<sup>2</sup> It should be reminded that any isolated black hole at late time after its formation is stationary.

these modes with negative energies are permanently excited and as we shall see later their state is described by a thermal density matrix. As for the particles created outside a black hole (external particles), only very small their number can penetrate the potential barrier and reach infinity. Namely those particles form the Hawking radiation of a black hole. All other external particles are reflected by the potential barrier and fall down into the black hole. During the time when they are still outside the horizon, the corresponding internal modes (which are described by a thermal density matrix) give the contribution to the black hole entropy.

In order to make the definition of the black hole entropy more concrete we assume that there exists a stationary black hole and denote by  $\hat{\rho}^{\text{init}}$  the density matrix describing in the Heisenberg representation the initial state of quantum fields propagating in its background. One may consider e.g. the in-vacuum state for a black hole evaporating in the vacuum, or the Hartle-Hawking state for a black hole in equilibrium with thermal radiation. For an exterior observer the system under consideration consists of two parts: a black hole and radiation outside of it. The state of radiation outside the black hole is described by the density matrix which is obtained from  $\hat{\rho}^{\text{init}}$  by averaging it over the states which are located inside the black hole and are invisible in its exterior

$$\hat{\rho}^{\text{rad}} = \text{Tr}^{\text{inv}} \hat{\rho}^{\text{init}}. \quad (4)$$

For an isolated black hole this density matrix  $\hat{\rho}^{\text{rad}}$  in particular describes its Hawking radiation at infinity. For a black hole in thermal equilibrium with radiation inside a cavity the density matrix  $\hat{\rho}^{\text{rad}}$  describes the state of thermal radiation.

Analogously we define the density matrix describing the state of a black hole as

$$\hat{\rho}^H = \text{Tr}^{\text{vis}} \hat{\rho}^{\text{init}}. \quad (5)$$

The trace-operators  $\text{Tr}^{\text{vis}}$  and  $\text{Tr}^{\text{inv}}$  in these relations mean that the trace is taken over the states located either outside ('visible') or inside ('invisible') the event horizon, correspondingly. We define the entropy of a black hole as

$$S^H = -\text{Tr}^{\text{inv}}(\hat{\rho}^H \ln \hat{\rho}^H). \quad (6)$$

The proposed definition of the entropy of a black hole is similar to the definition of the entropy of a usual black body. The definition is invariant in the following sense. Independent changes of vacuum definitions for 'visible' and 'invisible' states do not change the value of  $S^H$ . Bogolubov's transformations describing an independent changes of the vacuum states inside and outside the black hole can be represented by the unitary operator  $\hat{U} = \hat{U}^{\text{vis}} \otimes \hat{U}^{\text{inv}}$ , where  $\hat{U}^{\text{vis}}$  and  $\hat{U}^{\text{inv}}$  are unitary operators in the Hilbert spaces of 'visible' and 'invisible' particles, correspondingly. The above used trace operators are invariant under such transformations.

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The energy is defined with respect to the corresponding Killing vector. For a static black hole the Killing vector is spacelike inside the event horizon, which makes it possible the existence of the negative energy states.

In order to define the states one usually use modes expansion. The modes are characterized by a complete set of quantum numbers. Due to the symmetry properties one can choose such a subset  $J$  of quantum numbers connected with conservation laws (such as orbital and azimuthal angular momenta, helicity and so on) that guarantees the factorization of the density matrices. In the absence of mutual interaction of different fields the subset  $J$  necessarily includes also the parameters identifying the type of the field (e.g. mass, spin, and charge). The factorization in particular means that

$$\hat{\rho}^{\text{init}} = \otimes_J \hat{\rho}_J^{\text{init}}, \quad (7)$$

where  $\hat{\rho}_J^{\text{init}}$  is acting in the Hilbert space  $\mathcal{H}_J$  of states with the chosen quantum numbers  $J$ , while the complete Hilbert space is  $\mathcal{H} = \otimes_J \mathcal{H}_J$ . The factorization also means that the separation into ‘visible’ and ‘invisible’ states can be done independently in each subset of modes with a fixed  $J$  so that

$$\hat{\rho}^H = \otimes_J \hat{\rho}_J^H \quad \hat{\rho}_J^H = \text{Tr}_J^{\text{vis}} \hat{\rho}_J^{\text{init}}, \quad (8)$$

where all the operators with subscript  $J$  are acting in the Hilbert space  $\mathcal{H}_J$ <sup>3</sup>.

We illustrate the main steps of the calculations of the contribution of a given field to the density matrix of a black hole for a spherically-symmetric black hole. (The presence of charge and rotation does not create problems.) Moreover for simplicity we assume that a black hole is surrounded by a spherical mirror-like boundary of size  $r_B < 3M$ , so that the black hole is in thermal equilibrium with the radiation inside the cavity and the state of the system is described by the Hartle-Hawking vacuum state. The easiest way to introduce this state as well as to give definition of the ‘up’- and ‘side’-modes we use to describe the states of ‘visible’ and ‘invisible’ particles one can use the following useful trick proposed by Hawking in his original paper on the black hole evaporation [22]. At late time after the formation of a black hole the spacetime with the high accuracy is described by a stationary (in our case static) metric. For a given black hole we define its ‘eternal version’ as a spacetime of an eternal black hole which has the same global parameters (mass, charge, angular momentum) as the given black hole. Modes of the field  $\phi$  propagating at late-time in an ‘original’ black hole can be traced back in time in the spacetime of its ‘eternal version’ up to the initial global Cauchy surface  $\Sigma$  described by the equation  $t = 0$ , where  $t$  is the global time parameter defined by the Killing vector. ‘Up’-modes are defined as positive (in time  $t$ ) solutions vanishing in the left wedge  $R_-$ , and ‘side’-modes are defined as negative

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<sup>3</sup> Instead of non-normalizable monochromatic waves  $f_{\omega lm}$  with fixed frequency  $\omega$  it is convenient to consider normalized wave packets constructed from the monochromatic waves

$$f_{jnlm} = \delta^{-1/2} \kappa^{-1} \int_{j\delta}^{(j+1)\delta} \exp(2\pi i n \kappa^{-1} \omega / \delta) f_{\omega lm} d\omega,$$

where  $0 < \delta \ll 1$ . In what follows we shall be working with these type wavepackets and assume that the collective index  $J = (j, n, l, m)$ . We denote by  $\lambda$  the collective index  $\lambda = (\omega, l, m)$ .

(in time  $t$ ) solutions vanishing in the right wedge  $R_+$ . In the presence of mirror-like boundaries  $B$  and  $B'$  they are to satisfy the corresponding boundary conditions at  $B$  and  $B'^4$ .

By using the linear combinations of the operators of creation and annihilation for 'up'- and 'side'-particles one can construct the operators which annihilate the Hartle-Hawking vacuum  $|HH\rangle$ . For our choice of the initial state  $\hat{\rho}^{\text{init}} = |HH\rangle\langle HH|$ . This density matrix can be evidently expressed as the function of the 'up' and 'side' creation and annihilation operators. In order to separate the states into 'visible' and 'invisible' we consider a spacelike or null surface  $\Sigma_0$  which cross the event horizon at late time after black hole formation. An 'up'-particle is 'visible' if it crosses the surface  $\Sigma_0$  of a chosen moment of time outside the horizon. For this mode  $J$ , the density matrix  $\hat{\rho}_J^H$ , describing the state of the corresponding 'side'-particle, is obtained by tracing  $\hat{\rho}_J^{\text{init}}$  over the Hilbert space of 'up'-particles with index  $J$ . As the result one obtains the thermal density matrix of the form [16]

$$\hat{\rho}_J^H = \rho_J^0 \exp \left[ -(\omega_J/T_H) \hat{\alpha}_J^{SIDE} \hat{\alpha}_J^{SIDE} \right]. \quad (9)$$

In other words Hawking radiation of a black hole is accompanied by thermal excitation of 'side' modes (black hole's internal degrees of freedom). For the 'up'-modes  $J$  crossing  $\Sigma_0$  inside the horizon, both 'up' and 'side' particles are 'invisible'. We denote by  $\hat{\rho}_J^\bullet$ , the density matrix for such a pair. The density matrix  $\hat{\rho}_J^\bullet$  evidently describes the pure state, so that such a pair does not contribute to the entropy of a black hole. For the total density matrix  $\hat{\rho}^H$  of a black hole we have the following representation

$$\hat{\rho}^H = \prod_J' \hat{\rho}_J^H \prod_J'' \hat{\rho}_J^\bullet. \quad (10)$$

Here the prime indicates that the product includes only those states for which the 'up'-modes are 'visible', and double prime indicates that the product includes states for which the 'up'-modes are 'invisible'.

The expression (9) was obtained for the special choice of the initial state. For another choice of the initial state this expression must be modified. What is important that the modifications practically do not influence the expression for  $\hat{\rho}_J^H$  for late-time modes  $J$ . A 'side'-mode propagating inside a black hole close to the horizon at late-time being traced back in time reaches  $\mathcal{J}^-$  with a huge blue shift proportional to  $\exp \kappa t$ , where  $t$  is time past after the formation of the black hole. It means that in order to change the distribution for the mode  $J$  (e.g. to add an additional quantum in this state) one needs to send from  $\mathcal{J}^-$  the excitation which has exponentially large energy. At sufficiently late time this energy for given frequency is much larger than the mass of the black hole. For this reason the density matrix for 'side'-modes at late time will have the universal form (9).

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<sup>4</sup> For more details concerning the definition of 'up' and 'side' modes and their relations with other standard modes ('in', 'out', and 'down') defined in a black hole's exterior, see [20]

### 3 Statistical-Mechanical Entropy

By using the density matrix of a black hole  $\hat{\rho}^H$  one can calculate the corresponding statistical-mechanical entropy  $S^{SM} = -\text{Tr}(\hat{\rho}^H \ln \hat{\rho}^H)$ . The main contribution to the entropy of a black hole is given by 'side' modes of fields located in the very close vicinity of the horizon. Contributions of different fields enter  $S^{SM}$  additively. The factorization property (8) allows to write

$$S^{SM} = \sum_J S_J^{SM}, \quad S_J^{SM} = -\text{Tr}_J^{\text{inv}}(\hat{\rho}_J^H \ln \hat{\rho}_J^H), \quad (11)$$

where all the operators with subscript  $J$  are acting in the Hilbert space  $\mathcal{H}_J$ . In accordance with this decomposition one has<sup>5</sup>

$$S^{SM} = \sum_J' s(\beta\omega_J), \quad (12)$$

where

$$s(\beta\omega) = \frac{\beta\omega}{e^{\beta\omega} - 1} - \ln(1 - e^{-\beta\omega}), \quad (13)$$

is the entropy of a single oscillator with the frequency  $\omega$  at temperature  $T = 1/\beta$  and the prime in sum indicates that the summation is taken over the modes for which the corresponding 'up'-particle is 'visible' at the chosen moment of time  $\Sigma_0$ . The terms  $\hat{\rho}_J^\bullet$  corresponding to a pure state of a pair inside a black hole do not contribute to  $S^{SM}$ .

Returning from wave packets  $J$  to monochromatic waves  $\lambda$  one obtains the following result for the contribution of a chosen field to  $S^{SM}$

$$S^{SM} = \int d\mathbf{x} \sum_\lambda \mu_\lambda(\mathbf{x}) s(\beta\omega_\lambda). \quad (14)$$

Here

$$\mu_\lambda(\mathbf{x}) = g^{\tau\tau} g^{1/2} [R_\lambda(\mathbf{x})]^2 \quad (15)$$

is a phase space density of quantum modes and  $R_\lambda(\mathbf{x})$  are spatial harmonics corresponding to the mode with a collective quantum number  $\lambda = (\omega, l, m)$ .

The so defined  $S^{SM}$  contains a divergence, connected with the integration over the space regions near the horizon and is of the form [16]

$$S^{SM} \approx \frac{\alpha}{\varepsilon} \quad (16)$$

where  $\alpha$  is dimensionless parameter depending on the type of a field,  $\varepsilon = (l/r_+)^2$ , and  $l$  is the proper distance cut-off parameter. For a conformal scalar massless field

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<sup>5</sup> We use units in which  $G = c = \hbar = k = 1$ .

$\alpha = 1/90$ . For a fixed value  $l$  of the cut-off parameter, the expression for  $S^{SM}$  does not depend on the particular choice of the surface  $\Sigma_0$ , which was introduced to define 'visible' and 'invisible' particles. One may expect that quantum fluctuations of the horizon may provide natural cut-off and make  $S^{SM}$  finite. Simple estimations of the cut-off parameter show that  $l \approx l_P$  and  $S^{SM} \approx S^{BH}$  [16]<sup>6</sup>.

## 4 No-Boundary Wave Function

Another approach to the problem of dynamical degrees of freedom of a black hole was proposed in Ref.[20]. Its basic idea is the following. The study of propagation of perturbations of physical fields in the spacetime of a real black hole can be reduced to the analogous problem for its 'eternal version'. In particular one can trace back in time the perturbations in the space of the 'eternal version' until they cross the section  $\Sigma$  described by the equation  $t = 0$ . This section is known as the Einstein-Rosen bridge. The number and properties of the physical fields depends on the particular model. In any case the gravitational field must be included. The initial data at  $\Sigma$  for the gravitational perturbations at  $\Sigma$  can be related with small deformations of the geometry of the Einstein-Rosen bridge. The space of physical configurations of a system including a black hole can be related to the space of 'deformations' of the Einstein-Rosen bridge of the eternal black hole and possible configurations of other (besides the gravitational) fields on it, which obey the constraints and preserve asymptotic flatness. In a spacetime of an 'eternal version' of a black hole, perturbations with initial data located on the inner part of the Einstein-Rosen bridge ( $\Sigma_-$ ) are propagating to the future remaining entirely inside the horizon, and hence the corresponding perturbations in a 'physical' black hole also always remain under the horizon. That is why these data should be identified with internal degrees of freedom of a black hole. This construction allows natural generalization to the case when the deformations of the Einstein-Rosen bridge are not small. A quantum state of a black hole can be described by a wavefunction defined as a functional on the configuration space of deformations of the Einstein-Rosen bridge. In this representation deformations of the external ( $\Sigma_+$ ) and internal ( $\Sigma_-$ ) parts of the Einstein-Rosen bridge naturally represent degrees of freedom of matter outside the black hole and black hole's internal degrees of freedom.

The no-boundary ansatz (analogous to Hartle-Hawking ansatz in quantum cosmology) singles out a state which plays the role of a ground state of the system [20]. In order to describe this ansatz for a black hole we consider a half of the Hawking-Gibbons instanton, i.e. the space of the Euclidean black hole with the metric (24)

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<sup>6</sup> The loop expansion can be formulated as the expansion in powers of  $\hbar$ , and hence  $S^{SM}$ , which is one-loop quantity, must contain extra  $\hbar$  with respect to the tree-level contribution  $S^{BH}$ . Nevertheless the calculations show that  $S^{SM}$  and  $S^{BH}$  are of the same order of magnitude. It happens because the cut-off parameter needed to make  $S^{SM}$  finite is also dependent on  $\hbar$ :  $\varepsilon \sim \hbar$ .

and  $\tau \in (0, \pi/\kappa)$  Besides the boundary at infinity  $\partial\mathbf{M}_\infty$  this Euclidean space  $\mathbf{M}$  possesses only one boundary  $\partial\mathbf{M}$  which is isometric to the Einstein-Rosen bridge.

The no-boundary wavefunction of a black hole [20] is defined as a path integral

$$\Psi({}^3g(\mathbf{x}), \varphi(\mathbf{x})) = \int \mathcal{D}{}^4g \mathcal{D}\phi e^{-\mathbf{I}[{}^4g, \phi]} \quad (17)$$

of the exponentiated gravitational action  $\mathbf{I}[{}^4g, \phi]$  over Euclidean 4-geometries and matter-field configurations on those spacetime histories of physical fields  $\phi = \phi(x)$  on  $\mathbf{M}$  that generate the Euclidean 4-geometries asymptotically flat at the infinity  $\partial\mathbf{M}_\infty$  of spacetime and are subject to the conditions  $({}^3g(\mathbf{x}), \varphi(\mathbf{x}))$ ,  $\mathbf{x} \in \partial\mathbf{M}$ , – the collection of 3-geometry and boundary matter fields on  $\partial\mathbf{M}$ , which are just the argument of the wavefunction (17).  $\mathbf{I}[{}^4g, \phi]$  is the Lagrangian gravitational action in terms of these fields. The integration measure  $\mathcal{D}{}^4g \mathcal{D}\phi$  involves the local functional measure the structure of which is not very important for our purposes.

By its construction the no-boundary wavefunction of a black hole is symmetric with respect to the transposition of the interior and exterior parts of the Einstein-Rosen bridge. We call this property *duality*. For a 'real' black hole formed in the gravitational collapse, this exact symmetry is broken. Nevertheless, since there is a close relation between physics of a 'real' black hole and its 'eternal version', the duality of the above type plays an important role and allows one, for example, to explain why the approach based on identifying the dynamical degrees of freedom of a black hole with its external modes gives formally the same answer for the dynamical entropy of a black hole as our approach.

In the semiclassical approximation the no-boundary wavefunction Eq.(17) takes the form

$$\Psi(\varphi, M) = P e^{-2\pi M^2 - \mathbf{I}_2[\phi(\varphi)]}, \quad (18)$$

where  $\mathbf{I}_2[\phi(\varphi)]$  is a quadratic term of the action in the linearized physical fields, and  $\phi(\varphi)$  is a solution of the corresponding field equations on  $\mathbf{M}$  matching the boundary conditions  $\varphi$  on  $\partial\mathbf{M}$ . Denote by  $\varphi_{\lambda,\pm}$  the coefficients in the decomposition of the field  $\varphi(\mathbf{x})$  on  $\Sigma_\pm$  (an external and internal parts of the Einstein-Rosen bridge  $\Sigma = \partial\mathbf{M}$ ) in the basis of spatial harmonics  $\varphi_\pm(\mathbf{x}) = \sum_\lambda \varphi_{\lambda,\pm} R_\lambda(\mathbf{x})$ . Then the quadratic form  $\mathbf{I}_2[\phi(\varphi)]$  is

$$\mathbf{I}_2(\varphi_+, \varphi_-) = \frac{1}{2} \sum_\lambda \left\{ \frac{\omega_\lambda \cosh(\beta_0 \omega_\lambda / 2)}{\sinh(\beta_0 \omega_\lambda / 2)} (\varphi_{\lambda,+}^2 + \varphi_{\lambda,-}^2) - \frac{2\omega_\lambda}{\sinh(\beta_0 \omega_\lambda / 2)} \varphi_{\lambda,+} \varphi_{\lambda,-} \right\}. \quad (19)$$

The dependence of the no-boundary wavefunction on the mass  $M$  shows in this state the it is most probable to find a black hole of the smallest possible (Planckian) mass. For other states the dependence of the probability on the mass parameter is different. One might expect that for a state with fixed average value  $M_0$  of energy



(mass) the corresponding mass dependent part of the probability will have the form  $\exp[-(M - M_0)^2/2\Delta^2]$ , where  $\Delta$  is the mass dispersion.

For study the fields contribution to the statistical-mechanical entropy in the one-loop approximation it is sufficient to fix mass  $M$  of a black hole as a parameter in the wave function, and consider only the part describing fields perturbations. It is possible to show that for a fixed mass parameter  $M$  the part  $\exp -\mathbf{I}_2(\varphi_+, \varphi_-)$  of the no-boundary wave function (18) with  $\mathbf{I}_2(\varphi_+, \varphi_-)$  given by (19) describes the Hartle-Hawking vacuum state of the corresponding perturbations  $\phi$ .

By tracing over the external variables one obtains the density matrix of a black hole and can calculate its statistical-mechanical entropy [20]. The result coincides with (16). The reason why the calculations based on a 'real' black hole and on its 'eternal version' give the same result for the statistical-mechanical entropy of a black hole is the following. As it was already mentioned the late-time occupation of 'side'-modes does not depend of the particular choice of the initial state, so that to compare the results of the calculations one can choose the Hartle-Hawking state for the 'real' black hole, which is virtually the same as for the no-boundary wave function. On the other hand the main contribution to the entropy is connected with modes, propagating in the very close vicinity of the horizon, which are highly blue-shifted and for which the geometric-optics approximation works extremely well when one propagates these modes to the Einstein-Rosen bridge  $\Sigma$  in the eternal version of a black hole. For these reasons the counting of modes, contributing to the statistical-mechanical entropy for a 'real' black hole and for its 'eternal version' give the same answer.

## 5 Thermodynamical Entropy of a Black Hole

The Bekenstein-Hawking entropy by its definition coincides with the thermodynamical entropy of a black hole. We discuss now the relation between the thermodynamical and statistical-mechanical entropies and show that for a black hole these entropies are different.

In order to derive thermodynamical characteristics of a black hole it is convenient to begin with the partition function  $Z(\beta)$ . It is related with the free energy  $F$  ( $Z(\beta) = \exp(-\beta F)$ ) and is defined by the functional integral[11, 25]

$$Z(\beta) = \int D[g, \phi] \exp(iI[g, \phi]), \quad (20)$$

where  $I[g, \phi]$  is the action for the gravitational field  $g$  and some other fields  $\phi$ . The state of the system is determined by the choice of the boundary conditions on the metrics and fields that one integrates over. For the canonical ensemble describing the gravitational fields within a spherical box of radius  $r_B$  at temperature  $T_B$  one must integrate over all the metrics inside  $r_B$  which are periodically identified in the imaginary time direction with period  $\beta_B = T_B^{-1}$ . Denote by  $(g_0, \phi_0)$  a point of the

extremum of the action  $I[g, \phi]$ , then

$$\ln Z = iI[g_0, \phi_0] + \ln \int D[\bar{g}, \bar{\phi}] \exp(iI_2[\bar{g}, \bar{\phi}]) \quad (21)$$

where  $\bar{g}_{\mu\nu} = g_{\mu\nu} - g_{0\mu\nu}$ ,  $\bar{\phi} = \phi - \phi_0$ , and  $I_2[\bar{g}, \bar{\phi}] = I[g, \phi] - I[g_0, \phi_0]$  is quadratic in the perturbations  $\bar{g}$  and  $\bar{\phi}$ .

For vanishing background field  $\phi_0 = 0$  the extremum  $g_0$  is a solution of the vacuum Einstein equations. This solution for given boundary conditions coincides with the Euclidean black hole (a Hawking-Gibbons instanton). The corresponding metric can be obtained from the Schwarzschild solution

$$ds^2 = -Bdt^2 + B^{-1}dr^2 + r^2d\omega^2, \quad (22)$$

$$d\omega^2 = d\theta^2 + \sin^2\theta d\varphi^2, \quad B = 1 - 2M/r, \quad (23)$$

by the Wick's rotation of time  $t \rightarrow -i\tau$ . The metric of the Euclidean black hole is

$$ds^2 = Bd\tau^2 + B^{-1}dr^2 + r^2d\omega^2. \quad (24)$$

This metric is regular at the horizon  $r = r_+ = 2M$  (where  $B = 0$ ) only provided  $\tau$  is periodic with the period  $2\pi/\kappa$  ( $\kappa$  is the surface gravity of a black hole, for the Schwarzschild black hole  $\kappa = 1/4M$ ). The property of periodicity with respect to the imaginary time  $i\tau$  with the period  $\beta = 2\pi/\kappa$  implies in particular that a black hole is in thermal equilibrium with surrounding thermal radiation of the field  $\phi$ , provided the temperature of radiation, measured at infinity, is  $T_{BH} = \beta^{-1} = \kappa/2\pi$ .

For this extremum  $iI[g_0, \phi_0] = -I_E[g_0]$ , where  $I_E$  is the Euclidean action. The relation (21) implies that the free-energy  $F$  can be written as

$$F = F_0 + F_1 + \dots, \quad (25)$$

where  $F_0 = \beta^{-1}I_E[g_0]$  and  $F_1$  are the tree-level and one-loop contributions, respectively, and dots denote higher order terms in loops expansion.

We demonstrate now that the tree-level part of the free energy is directly connected with the Bekenstein-Hawking entropy, while the statistical-mechanical entropy is related to the one-loop contribution  $F_1$ . In what follows we assume that a black hole is in equilibrium with the thermal radiation inside the cavity of radius  $r_B$ . In the thermal equilibrium the mass  $M$  of a black hole is related to  $\beta$  as  $\beta = 4\pi r_+$ , where  $r_+ = 2M$  is the gravitational radius. The equilibrium is stable if  $r_B < 3r_+/2$ . The tree-level contribution of the black hole to the free energy of the system can be written in the form [12, 13]

$$F_0 = r_B \left( 1 - \sqrt{1 - r_+/r_B} \right) - \pi r_+^2 \beta_B^{-1}, \quad (26)$$

where  $\beta_B = 4\pi r_+(1 - r_+/r_B)^{1/2}$  is the inverse temperature at the boundary  $r_B$ . The tree-level contribution  $S_0^{TD}$  to the thermodynamical entropy of a black hole defined as

$S_0^{TD} = -\partial_{T_B} F_0 \equiv \beta_B^2 \partial_{\beta_B} F_0$  is  $S_0^{TD} = \pi r_+^2 \equiv A/4l_P^2 \equiv S^{BH}$ . In addition to this tree-level contribution which identically coincides with the Bekenstein-Hawking entropy  $S^{BH}$  there are also one-loop contributions directly connected with dynamical degrees of freedom of the black hole, describing its quantum excitations. We consider them now in more details.

By using Eq.(21) the one-loop contribution  $F_1$  to the free energy can be written in the form  $F_1 = -\beta^{-1} \ln Z_1$ , where

$$Z_1 = \int D[\varphi] \exp(-I_E[\varphi]), \quad (27)$$

and  $I_E[\varphi]$  is the quadratic Euclidean action of the field configuration  $\varphi \equiv (\bar{g}, \bar{\phi})$ . The integration is performed over all the perturbations fields  $\varphi$  that are real on the Euclidean section with metric (24) and are periodic in imaginary time coordinate  $\tau$  with period  $\beta^7$ . In the one-loop approximation different fields give independent contributions to  $F_1$ . For this reason it is sufficient to calculate the contribution of a chosen field  $\varphi$  and then add all the contributions corresponding to different fields. The integral (27) is ultraviolet divergent and requires regularization. The regularized value of  $F_1$  may depend on some regularization mass parameter  $\mu$ [14, 26]. Below we assume that the corresponding regularization is made and use the notation  $Z_1$  and  $F_1$  for the renormalized values of these quantities.

According to its definition the one-loop contribution  $S_1^{TD}$  to the thermodynamical entropy of a black hole is determined by the total response of the one-loop free energy  $F_1$  on the change of the temperature. Besides the direct dependence of  $F_1$  on temperature it also depends on the mass  $M$  of a black hole. In the thermal equilibrium  $M$  is a function of temperature. Thus we have ( $r_+ = 2M$ )

$$S_1^{TD} = \beta^2 \frac{dF_1}{d\beta} \equiv \beta^2 \left. \frac{\partial F_1}{\partial \beta} \right|_{r_+} + \beta^2 \left. \frac{\partial F_1}{\partial r_+} \right|_{\beta} \frac{dr_+}{d\beta}. \quad (28)$$

The first term in the right-hand-side of this relation is equal to the one-loop contribution  $S_1^{SM}$  to the statistical-mechanical entropy. In order to justify this claim we use the fact that the partition function  $Z_1$  is related to the thermodynamical partition function  $Z^T(\beta)$  of the canonical ensemble

$$Z^T(\beta) = \text{Tr } e^{-\beta \hat{H}} = \sum \exp(-\beta E_n), \quad (29)$$

where  $E_n$  is the energy (eigenvalue of the Hamiltonian  $\hat{H}$  of the field  $\varphi$ ). Namely, Allen[14] showed that  $F_1 \equiv -\beta^{-1} \ln Z_1$  differs from  $F^T \equiv -\beta^{-1} \ln Z^T$  only by terms

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<sup>7</sup> For a black hole located inside a cavity of radius  $r_B$  the field  $\varphi$  must obey also some boundary conditions at  $r_B$ . For our consideration dependence of  $F_1$  on  $r_B$  is not important. That is why we do not indicate it explicitly. We also use the inverse temperature at infinity  $\beta$  instead of  $\beta_B$ .

which are independent of  $\beta^8$ . Hence we have  $\beta^2(\partial F_1/\partial\beta)_{r_+} = \beta^2(\partial F^T/\partial\beta)_{r_+} = -\text{Tr}(\hat{\rho}\ln\hat{\rho}) \equiv S_1^{SM}$ , where  $\hat{\rho} = \exp[-\beta(\hat{H} - F^T)]$ . (We used here that the partial derivative  $(\partial/\partial\beta)_{r_+}$  with respect to the inverse temperature  $\beta$  commutes with Tr-operation.) The above relations allow one to rewrite Eq.(28) in the form

$$S_1^{TD} = S_1^{SM} + \Delta S_1, \quad (30)$$

where  $\Delta S_1 \equiv \beta^2(\partial F_1/\partial r_+)_{\beta} dr_+/d\beta$ . This relation shows that in order to obtain  $S_1^{TD}$  the statistical-mechanical entropy must be 'renormalized' by adding  $\Delta S_1$ . In particular, the relation (30) may give an explanation to the entropy renormalization procedure proposed by Thorne and Zurek[9].

For an investigation of  $\Delta S_1$ , it is convenient to rewrite  $F_1$  which enters the definition of  $\Delta S_1$  as  $F_1 = (F_1 - F^T) + F^T$ . The difference  $F_1 - F^T$  does not depend on  $\beta$  and hence one can calculate its value for zero temperature ( $\beta = \infty$ ). It indicates that the corresponding contribution to  $\Delta S_1$  is directly connected with vacuum polarization. The second contribution to  $\Delta S_1$  (connected with  $F^T$  term) arises because the complete derivative  $d/d\beta$ , defined by the relation (28), does not commute with Tr-operation<sup>9</sup>. It is instructive to demonstrate in more detail the origin of this non-commutation. The thermodynamical partition function  $Z^T$  in a static spacetime can be presented in the form[14]

$$Z^T = \prod_{\lambda} [1 - \exp(-\beta\omega_{\lambda})]^{-1}, \quad (31)$$

where  $\omega_{\lambda}$  are the energies of the single-particle states (or modes) and  $\lambda$  is the index enumerating these states. For the free energy  $F^T$  we have

$$F^T = \sum_{\lambda} f(\beta\omega_{\lambda}) = \int d\omega N(\omega|r_+) f(\beta\omega), \quad (32)$$

where  $f(\beta\omega) = \beta^{-1} \ln[1 - \exp(-\beta\omega)]$ , and  $N(\omega|r_+)$  is the density of number of states at the given energy  $\omega$  in a black hole of mass  $M = r_+/2$ .  $dN/d\beta \neq 0$ , since  $N(\omega|r_+)$  depends on the mass of a black hole<sup>10</sup>. This implies that  $d/d\beta$  and Tr-operation do not commute.

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<sup>8</sup> Recently the paper by D.V.Fursaev appeared as the preprint DSF-32/94 (hep-th/9408066), in which the problem of renormalization on the manifolds with cone-like singularity was considered. In this paper it was argued that in the presence of cone singularities additional temperature dependent divergencies may arise. If it happens the relation between  $Z_1$  and  $Z^T$  obtained by Allen[14] might be modified. As the result of this modification  $\Delta S_1$  in Eq.(30) would get extra contribution. However the main conclusions of the present paper remains unchanged. The reason is that for  $\beta = \beta_H$  the cone singularity disappears, the extra surface terms in the effective action vanish, and the renormalized free energy remains finite. So that the mechanism of compensation discussed later in the present paper remains valid.

<sup>9</sup> Possible non-commutation of differentiation with respect to temperature and Tr-operation for thermodynamical systems in the framework of thermofield approach was discussed in Ref.[24]

<sup>10</sup> It is easy to show that  $N(\omega|r_+) = \int d\mathbf{x} \sum_{\lambda} \delta(\omega - \omega_{\lambda}) |g^{tt} g^{1/2} R_{\lambda}(\mathbf{x})|^2$ , where  $R_{\lambda}(\mathbf{x})$  are spatial

## 6 Why the Entropy is $A/4$ ?

The calculation of the quantities which enter Eq.(30) is quite complicated. But important conclusions can be easily obtained by using some general properties of the free energy  $F_1$ . In general case (if  $\beta \neq \beta_H = 4\pi r_+$ ) the free energy  $F_1$  contains a divergence connected with the space integration over the region near the horizon. In order to regularize this divergence we suppose that the integration is performed up to the proper distance  $l$  to the horizon. Denote  $\varepsilon = (l/r_+)^2$ . In order to emphasize the dependence of  $F_1$  on the dimensionless cut-off parameter  $\varepsilon$  we shall write  $F_1 = F_1(\beta, r_+, \varepsilon)$ . The free energy has the same dimension as  $r_+^{-1}$  and hence it can be presented in the form  $F_1(\beta, r_+, \varepsilon) \equiv r_+^{-1} \mathcal{F}(\beta/\beta_H, \varepsilon)$ , where  $\mathcal{F}$  is dimensionless function of two dimensionless variables and  $\beta_H = 4\pi r_+ = 1/T_H$ , ( $T_H$  is the black hole temperature). The structure of the divergence near the Euclidean horizon can be analysed by using the curvature expansion of  $F_1$ . The leading divergent near the horizon  $r = r_+$  term is

$$\mathcal{F}(\beta/\beta_H, \varepsilon) \approx -\varepsilon^{-1} f(\beta/\beta_H). \quad (33)$$

An explicit form of the function  $f$  can be obtained by analyzing the free-energy in a flat cone space. The high-temperature expansion<sup>11</sup> (see, e.g. paper by Dowker and Kennedy[26]) shows that  $f(x) \sim x^{-4}$  for  $x \rightarrow \infty$ . The divergence of  $F_1$  at the Euclidean horizon for  $\beta \neq \beta_H$  reflects the fact that the number of modes that contribute to the free energy and entropy is infinitely growing as one considers regions closer and closer to the horizon [16]. For  $\beta = \beta_H$  the metric (24) is regular at the Euclidean horizon and hence the renormalized free energy calculated for the regular Euclidean manifold is finite. It implies that  $f(1) = 0$ .

The one-loop contribution of a quantum field  $\varphi$  to the statistical-mechanical entropy is

$$S_1^{SM} = \left[ \beta^2 \frac{\partial F_1}{\partial \beta} \right]_{\beta=\beta_H}. \quad (34)$$

It should be stressed that one must put  $\beta = \beta_H$  only after the differentiation. The leading (divergent near the horizon) term of  $S_1^{SM}$  is

$$S_1^{SM} \approx -\frac{4\pi}{\varepsilon} f'(1). \quad (35)$$

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harmonics[20]. The spatial integral is divergent near the horizon and requires cut-off. The main (leading at the horizon) part of  $N(\omega|r_+)$  can be calculated exactly. For example, for a scalar massless field  $N(\omega|r_+) \sim 8\omega^2 r_+^3 / (\pi\varepsilon)$ , where  $\varepsilon = (l/r_+)^2$  is a dimensionless cut-off parameter ( $l$  is a proper-distance cut-off).

<sup>11</sup> Dowker and Kennedy have shown that in the framework of this expansion the temperature  $T$  naturally enters in the combination  $T/\sqrt{|g_{tt}|}$ . For this reason the expansion can be used to get more detailed information about the behavior of the free-energy near the horizon.

This relation reproduces Eq.(16) with  $\alpha = -4\pi f'(1)$ . For a conformal massless scalar field  $f'(1) = -1/(360\pi)$ . If the proper-distance cut-off parameter  $l$  is of the order of the Planck length  $l_P$  then the contribution of the field to the statistical-mechanical entropy of a black hole is of the order  $S_1^{SM} \sim A/l_P^2$ , where  $A$  is the surface area of the black hole. In other words for the 'natural' choice of the cut-off parameter  $l \sim l_P$  the one-loop statistical-mechanical entropy  $S_1^{SM}$  of a black hole is of the same order of magnitude as the tree-level Bekenstein-Hawking entropy  $S^{BH}$ .

We show now that the additional term  $\Delta S_1$  in Eq.(30) always exactly compensates the divergence of  $S_1^{SM}$  at  $\varepsilon \rightarrow 0$ , so that  $S_1^{TD}$  remains finite in this limit. The key point of the proof is the above mentioned property of the renormalized free energy. Namely, for  $\beta = \beta_H = 4\pi r_+$  the point  $r = r_+$  is a regular point of the regular Euclidean manifold with metric (24) and hence the renormalized partition function  $Z_1$  calculated for any finite region of this manifold is finite. (We should recall that the black hole is surrounded by a boundary, so that the integration must be limited by  $r \leq r_B$ .) It implies that  $F_1$  calculated for  $\beta = \beta_H$  (i.e.,  $F_1 = r_+^{-1} \mathcal{F}(1, 0)$ ) is also finite. The one-loop contribution  $S_1^{TD}$  of a field to the thermodynamical entropy of a black hole can be obtained by differentiation of  $F_1$  with respect to the inverse temperature, provided one substitutes  $\beta_H = \beta$  into  $F_1$  *before* its differentiation. By using Eq.(30) one gets

$$S_1^{TD} = 4\pi\beta_H^2 \frac{\partial}{\partial\beta_H} \left( \frac{\mathcal{F}(1, \varepsilon)}{\beta_H} \right)_{\varepsilon=0} = 4\pi \left[ -\mathcal{F}(1, 0) + \left( \frac{\partial\mathcal{F}(1, \varepsilon)}{\partial\varepsilon} \frac{\partial\varepsilon}{\partial\ln\beta_H} \right)_{\varepsilon=0} \right]. \quad (36)$$

Because for  $\beta = \beta_H$  the free energy  $F_1$  does not contain divergence at the Euclidean horizon, the quantity in the square brackets of Eq.(36) is finite. It means that the additional contribution  $\Delta S_1$  exactly compensates the divergent terms of  $S_1^{SM}$ , so that the contribution  $S_1^{TD}$  of the quantum field  $\varphi$  to the thermodynamical entropy of a black hole is of order of  $O(\varepsilon^0)$ . In particular it means that  $S_1^{TD}$  is independent of the nature of the cut-off  $\varepsilon$ , which is assumed in  $S_1^{SM}$  and which for its calculation requires knowledge of physics at the Planckian scale. In other words, the thermodynamical entropy of a black hole is completely determined by low energy physics.  $S_1^{TD}$  contains the part which depends on  $r_B$ . This part describes the entropy of thermal gas of quanta of  $\varphi$  field, located outside a black hole within the cavity of size  $r_B$ . In addition  $S_1^{TD}$  also contains part independent of  $r_B$  describing quantum corrections to the black hole entropy. For black holes of mass much larger than the Planckian mass these corrections are much smaller than  $A/l_P^2$  and can be neglected. As the result of the above described compensation mechanism the dynamical degrees of freedom of the black hole practically do not contribute to its thermodynamical entropy  $S^{TD}$ , and the latter is defined by the tree-level quantity  $S^{BH}$ .

To make the basic idea clearer we restricted ourselves in the above discussion by considering a non-rotating black hole. The analysis is easily applied to the case of a charged rotating black hole as well as to their non-Einsteinian and n-dimensional generalizations. It is interesting that for black holes in the generalized gravitational

theories the thermodynamical and statistical-mechanical entropy may have different dependence on the mass  $M$  of a black hole. In particular[27] for a two dimensional dilaton black hole  $S^{BH} = 4\pi M/\sqrt{\lambda}$ , while  $S_1^{SM} \sim \ln \varepsilon$ .

To summarize it has been shown that the Bekenstein-Hawking entropy does not coincide with the statistical-mechanical entropy  $S_1^{SM} = -\text{Tr}(\hat{\rho} \ln \hat{\rho})$  of a black hole. The latter entropy is determined by internal degrees of freedom of the black hole, describing different states which may exist inside a black hole for the same value of its external parameters. The discrepancy arises because in the state of thermal equilibrium the parameters of internal degrees of freedom of a black hole depend on the temperature of the system in the universal way. This results in the universal cancellation of all those contributions to the thermodynamical entropy which depend on the particular properties and number of fields. That is why the thermodynamical entropy of black holes in Einstein's theory is always  $S^{BH}$ .

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